

FACTORIALITY AND THE CONNES INVARIANT $T(\mathcal{M})$ FOR FREE PRODUCTS OF VON NEUMANN ALGEBRAS

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ABSTRACT. Sufficient conditions for factoriality are given for free products of von Neumann algebras with respect to states that are not necessarily traces. The Connes T -invariant of the free product algebra is found, which has implications for the type of the algebra. Roughly speaking, the free product of von Neumann algebras with respect to states, one of which is not a trace, is a type III factor.

Introduction. In this paper, we examine free products of (not necessarily injective) von Neumann algebras with respect to states that, in contrast to the cases studied in [D3], need not be traces. Our main theorem gives sufficient conditions for the free product to be a factor, and using it we are also able to make conclusions about the type of the free product factor. Similarly to the results of [D3], for factoriality of the free product it suffices that the original algebras not be too “lumpy,” *i.e.* not have big minimal projections. However, in contrast to [D3], the sufficient conditions given here are far from necessary for factoriality. Connes’ T -invariant is computed for the free product factor in terms of the original algebras and states, and this allows one to determine, in the case of separable preduals, if the free product factor is type III, and to gain information about its classification as type III_λ . Roughly speaking, the pattern for the type classification obtained is that as long as the states with respect to which we take the free product are not all traces, we get a type III factor, and it is type III_λ , where we are able to specify λ to within a choice of two values, one of which is 0.

Since the appearance of Voiculescu’s free probability theory in the mid ’80’s, (see [VDN] and references contained therein), and more specifically since Voiculescu’s random matrix model for freeness [V3], there has been steady progress in understanding the free group factors and free products of finite von Neumann algebras, [V2,R1,R2,R3,R4,D1,D2,D3]. For example, in [D3] the free product of any two finite injective von Neumann algebras was found and expressed in terms of the interpolated free group factors [D2], (see also [R3]), via “free dimension.”

More recently, in [R4] F. Rădulescu exhibited a 1-parameter group of trace-scaling automorphisms of $L(\mathbf{F}_\infty) \otimes B(\mathcal{H})$. In [R5], he showed that if ψ_λ is the state on the 2×2 matrices $M_2(\mathbf{C})$ defined with respect to a system of matrix units $\{e_{\iota j} \mid \iota, j = 1, 2\}$ by $\psi_\lambda(e_{12}) = 0$, $\psi_\lambda(e_{11}) = \frac{1}{1+\lambda}$ and $\psi_\lambda(e_{22}) = \frac{\lambda}{1+\lambda}$ for some $0 < \lambda < 1$, then the free product of a diffuse abelian von Neumann algebra with $(M_2(\mathbf{C}), \psi_\lambda)$ is a type III_λ factor having a discrete decomposition given by the appropriate subgroup of the 1-parameter group from [R4].

Note also the related work of L. Barnett [B].

One of A. Connes’ great achievements is his classification of type III factors in terms of trace scaling automorphisms of type II_∞ factors [C]. The type III factors are differentiated as type III_λ , ($0 \leq \lambda \leq 1$). Recall that his

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invariant $S(\mathcal{M})$ is the intersection of the spectra of the modular operators Δ_ϕ , as ϕ ranges over the set of all normal, semifinite, faithful (n.s.f.) weights on \mathcal{M} . The Connes classification is then that

$$\mathcal{M} \text{ is type } \begin{cases} \text{I or II} & \text{if } S(\mathcal{M}) = \{1\} \\ \text{III}_0 & \text{if } S(\mathcal{M}) = \{0, 1\} \\ \text{III}_\lambda & \text{if } S(\mathcal{M}) = \{0\} \cup \{\lambda^n \mid n \in \mathbf{Z}\}, 0 < \lambda < 1 \\ \text{III}_1 & \text{if } S(\mathcal{M}) = [0, +\infty). \end{cases}$$

Connes defined another invariant $T(\mathcal{M}) = \{t \in \mathbf{R} \mid \sigma_t^\phi \in \text{Int}\mathcal{M}\}$, where $(\sigma_t^\phi)_{t \in \mathbf{R}}$ is the modular automorphism group of \mathcal{M} corresponding to the n.s.f. weight ϕ , and $\text{Int}\mathcal{M}$ is the group of inner automorphisms of \mathcal{M} . He proved that this set is independent of ϕ and also that if $t \in T(\mathcal{M})$, then there is an n.s.f. weight ψ such that $\sigma_t^\psi = id$. This invariant $T(\mathcal{M})$ is related to the type classification of factors as follows:

$$T(\mathcal{M}) = \begin{cases} \mathbf{R} & \text{if } \mathcal{M} \text{ is type I or II} \\ \{0\} & \text{if } \mathcal{M} \text{ is type III}_1 \\ \frac{2\pi}{\ln \lambda} \mathbf{Z} & \text{if } \mathcal{M} \text{ is type III}_\lambda, \lambda \in (0, 1). \end{cases}$$

If a von Neumann algebra \mathcal{M} has separable predual, then $T(\mathcal{M}) = \mathbf{R}$ if and only if \mathcal{M} is semifinite, (see [S], 27.2). Thus for a given factor \mathcal{M} with separable predual, knowing $T(\mathcal{M})$ allows one to decide if \mathcal{M} is type III, and if so to specify the λ -classification to within a choice of at most two, one of which is III_0 .

Definition 0. Let ϕ be an n.s.f. weight on a von Neumann algebra \mathcal{M} , and let $(\sigma_t^\phi)_{t \in \mathbf{R}}$ be the resulting modular automorphism group of \mathcal{M} . We define

$$I(\mathcal{M}, \phi) = \{t \in \mathbf{R} \mid \sigma_t^\phi = id\},$$

which is clearly an additive subgroup of $T(\mathcal{M})$. Note that since $t \mapsto \sigma_t^\phi$ is continuous in the pointwise-strong topology on $\text{Aut}(\mathcal{M})$, we have that $I(\mathcal{M}, \phi)$ is a closed subgroup of \mathbf{R} .

Our main theorem shows that for “most” families of von Neumann algebras with states $((\mathcal{M}_\iota, \phi_\iota))_{\iota \in I}$, if $(\mathcal{M}, \phi) = \bigstar_{\iota \in I} (\mathcal{M}_\iota, \phi_\iota)$ is their free product, then $T(\mathcal{M}) = I(\mathcal{M}, \phi) = \bigcap_{\iota \in I} I(\mathcal{M}_\iota, \phi_\iota)$.

In §1, we prove a certainly well known result about the modular automorphism group of a free product. In §2 we define a technical quantity, called the expansion factor of a von Neumann algebra with a specified state, prove some lemmas about this quantity and prove our main theorem. In §3 the expansion factors of several von Neumann algebras are calculated, allowing one to decide when the hypotheses of our main theorem are satisfied. In §4 appear examples of specific results that follow from the main theorem, and also some limitations of the main theorem.

§1. The modular automorphism group of a free product

Let us review the construction of the free product of von Neumann algebras [V1] and then show that the modular automorphism (with respect to the free product state) is the free product of the modular automorphisms of the original algebras. This result has surely been known since Voiculescu in [V1] found the modular operator for a free product algebra. Let $(\mathcal{M}_\iota, \phi_\iota)_{\iota \in I}$ be a family of von Neumann algebras with specified faithful normal states. Our notation for the GNS construction is $\mathcal{H}_\iota = L^2(\mathcal{M}_\iota, \phi_\iota)$ with $\mathcal{M}_\iota \ni x \mapsto \hat{x} \in \mathcal{H}_\iota$, and the distinguished vector $\hat{1}$ is denoted by ξ_ι . The Hilbert algebra $\mathfrak{A}_\iota = (\mathcal{M}_\iota)^\wedge = \mathcal{M}_\iota \xi_\iota$ is dense in \mathcal{H}_ι , and we have the left representation π_ι of \mathcal{M}_ι on \mathcal{H}_ι given by $\pi_\iota(a)\hat{x} = (ax)^\wedge$. The modular operator, Δ_ι on \mathcal{H}_ι , is the unbounded positive operator constructed in the following way (see [T,SZ]). The conjugation operator, S_ι , is the closure of the unbounded operator defined by $S_\iota(\hat{x}) = (x^*)^\wedge$. Then $\Delta_\iota = S_\iota^* S_\iota$. Note that $\Delta_\iota \xi_\iota = \xi_\iota$.

Consider the free product of Hilbert spaces (with distinguished vectors) $(\mathcal{H}, \xi) = \bigstar_{\iota \in I} (\mathcal{H}_\iota, \xi_\iota)$, given by

$$\mathcal{H} = \mathbf{C}\xi \oplus \bigoplus_{n \geq 1} \left(\bigoplus_{(\iota_1 \neq \iota_2 \neq \dots \neq \iota_n)} \overset{\circ}{\mathcal{H}}_{\iota_1} \otimes \dots \otimes \overset{\circ}{\mathcal{H}}_{\iota_n} \right), \quad (1)$$

where $\overset{\circ}{\mathcal{H}}_\iota$ is the orthocomplement of ξ_ι in \mathcal{H}_ι . We often identify \mathcal{H}_ι with $\mathbf{C}\xi \oplus \overset{\circ}{\mathcal{H}}_\iota \subseteq \mathcal{H}$. There are injective, unital, normal $*$ -homomorphisms $\lambda_\iota : \mathcal{M}_\iota \rightarrow B(\mathcal{H})$, where $\lambda_\iota(a)$ acts essentially “on the first component” of each tensor product in (1). To be precise, for $x \in \ker \phi_\iota$ and $\zeta_1 \otimes \cdots \otimes \zeta_n \in \overset{\circ}{\mathcal{H}}_{\iota_1} \otimes \cdots \otimes \overset{\circ}{\mathcal{H}}_{\iota_n}$ we have

$$\lambda_\iota(x)\xi = \hat{x}$$

$$\lambda_\iota(x)(\zeta_1 \otimes \cdots \otimes \zeta_n) = \begin{cases} c\zeta_2 \otimes \cdots \otimes \zeta_n + (\pi_\iota(x)\zeta_1 - c\xi_\iota) \otimes \zeta_2 \otimes \cdots \otimes \zeta_n & \text{if } \iota_1 = \iota, \text{ where } c = \langle \pi_\iota(x)\zeta_1, \xi_1 \rangle, \\ \hat{x} \otimes \zeta_1 \otimes \cdots \otimes \zeta_n & \text{if } \iota_1 \neq \iota, \end{cases} \quad (2)$$

and of course in the first case of the second equation above, if $n = 1$ we take $c\xi + (\pi_\iota(x)\zeta_1 - c\xi_\iota)$. Then the free product von Neumann algebra is $\mathcal{M} = (\bigcup_{\iota \in I} \lambda_\iota(\mathcal{M}_\iota))''$ and has free product state $\phi(x) = \langle x\xi, \xi \rangle$. We write $(\mathcal{M}, \phi) = \bigstar_{\iota \in I} (\mathcal{M}_\iota, \phi_\iota)$. For ease of notation we often suppress the λ_ι and simply think of $\mathcal{M}_\iota \subset \mathcal{M}$. We sometimes write $x = \pi(x)$ for $x \in \mathcal{M}$ acting on \mathcal{H} , to emphasize the fact that \mathcal{M} acts “on the left.” Then $\mathcal{H} = L^2(\mathcal{M}, \phi)$ and $\mathfrak{A} = \mathcal{M}^\wedge = \mathcal{M}\xi$ is a Hilbert algebra containing the dense subspace

$$\mathbf{C}\xi \oplus \bigoplus_{n \geq 1} \left(\bigoplus_{(\iota_1 \neq \iota_2 \neq \cdots \neq \iota_n)} \overset{\circ}{\mathfrak{A}}_{\iota_1} \otimes \cdots \otimes \overset{\circ}{\mathfrak{A}}_{\iota_n} \right),$$

where $\overset{\circ}{\mathfrak{A}}_\iota = \mathfrak{A}_\iota \cap \overset{\circ}{\mathcal{H}}_\iota$. Now

$$\overset{\circ}{\mathfrak{A}}_{\iota_1} \otimes \cdots \otimes \overset{\circ}{\mathfrak{A}}_{\iota_n} = \overset{\circ}{\mathcal{M}}_{\iota_1} \cdots \overset{\circ}{\mathcal{M}}_{\iota_n} \xi,$$

where $\overset{\circ}{\mathcal{M}}_{\iota_k} = \ker \phi_{\iota_k}$, so the conjugation operator, S , for \mathfrak{A} acts on $\zeta_1 \otimes \cdots \otimes \zeta_n \in \overset{\circ}{\mathfrak{A}}_{\iota_1} \otimes \cdots \otimes \overset{\circ}{\mathfrak{A}}_{\iota_n}$ by

$$S(\zeta_1 \otimes \cdots \otimes \zeta_n) = (S_{\iota_n} \zeta_n) \otimes \cdots \otimes (S_{\iota_1} \zeta_1).$$

Thus

$$S^*(\zeta_1 \otimes \cdots \otimes \zeta_n) = (S_{\iota_n}^* \zeta_n) \otimes \cdots \otimes (S_{\iota_1}^* \zeta_1)$$

and so

$$\Delta(\zeta_1 \otimes \cdots \otimes \zeta_n) = (\Delta_{\iota_1} \zeta_1) \otimes \cdots \otimes (\Delta_{\iota_n} \zeta_n), \quad (3)$$

(this was first observed in [V1]).

Theorem 1. *Let $(\mathcal{M}, \phi) = \bigstar_{\iota \in I} (\mathcal{M}_\iota, \phi_\iota)$ be as above and let us write $\mathcal{M}_\iota \subset \mathcal{M}$. Let $(\sigma_t^{\phi_\iota})_{t \in \mathbf{R}}$ be the modular automorphism group of \mathcal{M}_ι with respect to the state ϕ_ι and let $(\sigma_t)_{t \in \mathbf{R}}$ be the modular automorphism group of \mathcal{M} with respect to the free product state ϕ . Then $\sigma_t(x) = \sigma_t^{\phi_\iota}(x) \forall x \in \mathcal{M}_\iota$, which of course determines σ_t .*

Proof. The modular automorphism is defined by $\sigma_t(x) = \Delta^{-it} x \Delta^{it} \forall x \in \mathcal{M}$ and $\sigma_t^{\phi_\iota}(x) = \Delta_\iota^{-it} x \Delta_\iota^{it} \forall x \in \mathcal{M}_\iota$. Writing now the injections $\lambda_\iota : \mathcal{M}_\iota \rightarrow \mathcal{M}$ to be absolutely clear, we must show that $\sigma_t(\lambda_\iota(x)) = \lambda_\iota(\sigma_t^{\phi_\iota}(x)) \forall x \in \mathcal{M}_\iota$. Since every modular automorphism sends 1 to 1, we can concentrate on the case $x \in \overset{\circ}{\mathcal{M}}_\iota$. But using (2) and (3) gives $\Delta^{-it} \lambda_\iota(x) \Delta^{it} \xi = \Delta^{-it} \lambda_\iota(x) \xi = \Delta^{-it} \hat{x} = \Delta_\iota^{-it} \hat{x} = \Delta_\iota^{-it} x \xi_\iota = \Delta_\iota^{-it} x \Delta_\iota^{it} \xi_\iota = \sigma_t^{\phi_\iota}(x) \xi_\iota = (\sigma_t^{\phi_\iota}(x))^\wedge = \lambda_\iota(\sigma_t^{\phi_\iota}(x)) \xi$. But two elements of \mathcal{M} that agree on ξ must be equal. \square

§2 Factoriality and the T -invariant of a free product.

Let (\mathcal{M}, ϕ) be a von Neumann algebra with normal, faithful (n.f.) state, let $\mathcal{H} = L^2(\mathcal{M}, \phi)$ have distinguished vector $\xi = \hat{1}$ and set $\overset{\circ}{\mathcal{H}} = \mathcal{H} \ominus \mathbf{C}\xi$. We will denote by $\overline{\mathcal{H}}$ the dual Hilbert space of \mathcal{H} , which is just \mathcal{H} with conjugate scalar multiplication. P_ξ will denote the orthogonal projection of \mathcal{H} onto $\mathbf{C}\xi$ and $P_{\overset{\circ}{\mathcal{H}}} = I - P_\xi$ the orthogonal projection of \mathcal{H} onto $\overset{\circ}{\mathcal{H}}$. \mathcal{M} acts by bounded operators on the left of \mathcal{H} by $\pi(a)\hat{x} = (ax)^\wedge \forall a, x \in \mathcal{M}$. Hence there is the dual anti-representation $\overline{\pi}$ of \mathcal{M} on $\overline{\mathcal{H}}$ given by

$$\langle \zeta, \overline{\pi}(x)\eta \rangle = \langle \pi(x)\zeta, \eta \rangle \forall x \in \mathcal{M}, \zeta \in \mathcal{H}, \eta \in \overline{\mathcal{H}}.$$

Of course, viewing elements of $\overline{\mathcal{H}}$ as being elements of \mathcal{H} , we have $\overline{\pi}(x)\eta = \pi(x^*)\eta$.

Definition 2.1. Recall $\mathcal{H} \otimes \mathcal{H}$ is equal to the space of Hilbert–Schmidt operators from $\overline{\mathcal{H}}$ into \mathcal{H} . Consider a bounded linear map $\alpha : \overline{\mathcal{H}} \rightarrow \mathcal{M}$. For $v \in \mathcal{H}$ we thus have

$$\pi(\alpha(\cdot))v : \overline{\mathcal{H}} \rightarrow \mathcal{H}. \quad (4)$$

Define $\mathcal{H} \otimes \mathcal{M}$ to be the space of α such that the operator in (4) is Hilbert–Schmidt, *i.e.* $\pi(\alpha(\cdot))v \in \mathcal{H} \otimes \mathcal{H}$, for every $v \in \mathcal{H}$, and furthermore that the norm of $\pi(\alpha(\cdot))v$ in $\mathcal{H} \otimes \mathcal{H}$, (*i.e.* its Hilbert–Schmidt norm), is uniformly bounded for $\|v\| = 1$. We then say that $\|\alpha\|_{HS} = \sup_{\|v\|=1} \|\pi(\alpha(\cdot))v\|_{HS}$. Set $\hat{\alpha} = \pi(\alpha(\cdot))\xi \in \mathcal{H} \otimes \mathcal{H}$. For $\zeta \in \mathcal{H}$ and $x \in \mathcal{M}$ the simple tensor $\zeta \otimes x \in \mathcal{H} \otimes \mathcal{M}$ is given by $(\zeta \otimes x)(\eta) = \langle \zeta, \eta \rangle x$. Note that for $v \in \mathcal{H}$ we have $\pi((\zeta \otimes x)(\cdot))v = \zeta \otimes (\pi(x)v) \in \mathcal{H} \otimes \mathcal{H}$.

A bounded linear map $\alpha : \overline{\mathcal{H}} \rightarrow \mathcal{M}$ is said to be *left–right \mathcal{M} –equivariant* if

$$\alpha(\pi(a)\eta) = \alpha(\eta)a \quad \forall a \in \mathcal{M}, \eta \in \overline{\mathcal{H}}.$$

The *expansion factor* of (\mathcal{M}, ϕ) , denoted $\text{ef}(\mathcal{M}, \phi)$, is defined to be the largest constant $c \geq 0$ such that

$$\|(P_{\mathcal{H}}^{\circ} \otimes P_{\mathcal{H}}^{\circ})\hat{\alpha}\| \geq c|\langle \hat{\alpha}, \xi \otimes \xi \rangle| \quad (5)$$

for every left–right \mathcal{M} –equivariant $\alpha \in \mathcal{H} \otimes \mathcal{M}$. Of course, if the right hand side of (5) is always zero, we have $\text{ef}(\mathcal{M}, \phi) = +\infty$.

Remark 2.2. The definitions above become somewhat more transparent, and the expansion factor easier to calculate, in the case where \mathcal{M} is finite dimensional. For then every linear map $\alpha : \overline{\mathcal{H}} \rightarrow \mathcal{M}$ is a sum of simple tensors, hence $\in \mathcal{H} \otimes \mathcal{M}$. The definition of left–right \mathcal{M} –equivariance can be rephrased in terms of $\hat{\alpha}$ only. For $a \in \mathcal{M}$, let $\rho(a)$ act “on the right” of \mathcal{H} , by $\rho(a)\hat{b} = (ba)^{\wedge} \forall b \in \mathcal{M}$. Then α is left–right \mathcal{M} –equivariant if and only if $(\pi(a) \otimes 1)\hat{\alpha} = (1 \otimes \rho(a))\hat{\alpha} \forall a \in \mathcal{M}$. So let us call $X \in \mathcal{H} \otimes \mathcal{H}$ left–right \mathcal{M} –equivariant if $(\pi(a) \otimes 1)X = (1 \otimes \rho(a))X \forall a \in \mathcal{M}$. Then $\text{ef}(\mathcal{M}, \phi)$ equals the infimum of $\|(P_{\mathcal{H}}^{\circ} \otimes P_{\mathcal{H}}^{\circ})X\|$ such that $X \in \mathcal{H} \otimes \mathcal{H}$ is left–right \mathcal{M} –equivariant and $\langle X, \xi \otimes \xi \rangle = 1$. We will use this to calculate expansion factors in §3.

Lemma 2.3. *Let \mathcal{M} be a diffuse von Neumann algebra, *i.e.* having no nonzero minimal projections, and let ϕ be any n.f. state on \mathcal{M} . Then every left–right \mathcal{M} –equivariant element of $\mathcal{H} \otimes \mathcal{M}$ equals zero, so $\text{ef}(\mathcal{M}, \phi) = +\infty$.*

Proof. Let $\alpha \in \mathcal{H} \otimes \mathcal{M}$ be left–right \mathcal{M} –equivariant, and let \mathcal{A} be a maximal abelian subalgebra (MASA) in \mathcal{M} . It is well known that there is a conditional expectation of norm 1, $E : \mathcal{M} \rightarrow \mathcal{A}$ (see [S], 10.15). Let $\mathcal{H}_{\mathcal{A}} = L^2(\mathcal{A}, \phi|_{\mathcal{A}}) \subseteq \mathcal{H}$ and $E\alpha = E \circ \alpha|_{\overline{\mathcal{H}}_{\mathcal{A}}} : \overline{\mathcal{H}}_{\mathcal{A}} \rightarrow \mathcal{A}$. Although $E\alpha$ is a bounded linear function and $\|E\alpha\| \leq \|\alpha\|$, in general $E\alpha \notin \mathcal{H}_{\mathcal{A}} \otimes \mathcal{A}$. However $E\alpha$ is left–right \mathcal{A} –equivariant. Indeed, we have for $a \in \mathcal{A}$ and $\eta \in \mathcal{H}_{\mathcal{A}}$ that $(E\alpha)(\pi(a)\eta) = E(\alpha(\pi(a)\eta)) = E(\alpha(\eta)a) = E(\alpha(\eta))a = (E\alpha(\eta))a$. We will show that $E\alpha$ must be zero. Let us take $\mathcal{A} = L^{\infty}([0, 1])$, $\mathcal{H}_{\mathcal{A}} = L^2([0, 1])$ with respect to Lebesgue measure, and denote $E\alpha = T$. Thus $T : \overline{L^2([0, 1])} \rightarrow L^{\infty}([0, 1])$ is bounded linear. The fact that $E\alpha$ is left–right \mathcal{A} –equivariant gives that

$$T(fg) = \overline{f}T(g) \quad \forall f \in L^{\infty}([0, 1]), g \in L^2([0, 1]).$$

So T is determined by its value at the constant function 1. Suppose for contradiction that $T \neq 0$. Then $T(1) \neq 0$. Let $\delta > 0$ and $S \subseteq [0, 1]$ be measurable of nonzero measure such that $|T(1)(t)| > \delta \forall t \in S$. Without loss of generality we may suppose $S \supseteq [0, R]$, some $0 < R \leq 1$. For $n \geq 1$ let $f_n \in L^{\infty}([0, 1])$ be

$$f_n(t) = \begin{cases} t^{-1/4} & \text{if } \frac{1}{n} \leq t \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then $\|f_n\|_{L^2}^2 = \int_{1/n}^1 t^{-1/2} dt < 2$. But $\|T(f_n)\|_{L^{\infty}} = \|f_n T(1)\|_{L^{\infty}} \geq n^{1/4} \delta$ for $n \geq \frac{1}{R}$. This contradicts the boundedness of T . So T must equal 0.

Hence we have shown that for every MASA $\mathcal{A} \subseteq \mathcal{M}$ and every conditional expectation $E : \mathcal{M} \rightarrow \mathcal{A}$ we must have $E \circ \alpha|_{\overline{\mathcal{H}}_{\mathcal{A}}} = 0$. Suppose for contradiction that $\alpha \neq 0$. Then since $\pi(\mathcal{M})\xi$ is dense in $\overline{\mathcal{H}}$ and since α is left-right \mathcal{M} -equivariant, we must have $\alpha(\xi) = x \neq 0$. Suppose the real part of x is nonzero, $x_{\text{Re}} = \frac{x+x^*}{2} \neq 0$, let \mathcal{A} be a MASA in \mathcal{M} that contains all the spectral projections of x_{Re} and let $E : \mathcal{M} \rightarrow \mathcal{A}$ be a conditional expectation. Thus $E\alpha(\xi) = E(x) = x_{\text{Re}} + iE(x_{\text{Im}})$. Since E is positive we know that $E(x_{\text{Im}})$ is self-adjoint and hence that $E(x) \neq 0$. This contradicts that $E\alpha = 0$. A similar argument works if $x_{\text{Im}} \neq 0$. \square

Lemma 2.4. *Let (\mathcal{M}, ϕ) be a von Neumann algebra with n.f. state and suppose that $1 \in \mathcal{N} \subseteq \mathcal{M}$ is a W^* -subalgebra and $E : \mathcal{M} \rightarrow \mathcal{N}$ is a projection of norm 1 satisfying $\phi \circ E = \phi$. Then $\text{ef}(\mathcal{M}, \phi) \geq \text{ef}(\mathcal{N}, \phi|_{\mathcal{N}})$.*

Proof. Let $\mathcal{H}_{\mathcal{N}} = L^2(\mathcal{N}, \phi|_{\mathcal{N}}) \subseteq \mathcal{H}$ and $P_{\mathcal{H}_{\mathcal{N}}}$ the orthogonal projection of \mathcal{H} onto $\mathcal{H}_{\mathcal{N}}$. Since E preserves ϕ it follows that $(Ex)^{\wedge} = P_{\mathcal{H}_{\mathcal{N}}}(\hat{x}) \forall x \in \mathcal{M}$. Suppose $\alpha \in \mathcal{H} \otimes \mathcal{M}$ is left-right \mathcal{M} -equivariant. Then letting $E\alpha = E \circ \alpha|_{\overline{\mathcal{H}}_{\mathcal{N}}} : \overline{\mathcal{H}}_{\mathcal{N}} \rightarrow \mathcal{N}$ we have as in the proof of Lemma 2.3 that $E\alpha$ is left-right \mathcal{N} -equivariant. We would like to show that for $E\alpha \in \mathcal{H}_{\mathcal{N}} \otimes \mathcal{N}$. First note that for $x \in \mathcal{M}$ we have $E(x)|_{\mathcal{H}_{\mathcal{N}}} = (P_{\mathcal{H}_{\mathcal{N}}} \circ x)|_{\mathcal{H}_{\mathcal{N}}}$. Hence for $v \in \mathcal{H}_{\mathcal{N}}$, $(E\alpha)(\cdot)v = (P_{\mathcal{H}_{\mathcal{N}}} \circ \alpha(\cdot))v$, so

$$((E\alpha)(\cdot)v)|_{\overline{\mathcal{H}}_{\mathcal{N}}} = (P_{\mathcal{H}_{\mathcal{N}}} \otimes P_{\mathcal{H}_{\mathcal{N}}})(\alpha(\cdot)v)|_{\overline{\mathcal{H}}_{\mathcal{N}}}) \in \mathcal{H}_{\mathcal{N}} \otimes \mathcal{H}_{\mathcal{N}}. \quad (6)$$

Thus $E\alpha \in \mathcal{H}_{\mathcal{N}} \otimes \mathcal{N}$ and $\|E\alpha\|_{HS} \leq \|\alpha\|_{HS}$. A particular case of (6) is $(E\alpha)^{\wedge} = (P_{\mathcal{H}_{\mathcal{N}}} \otimes P_{\mathcal{H}_{\mathcal{N}}})\hat{\alpha}$. Since $E\alpha$ is left-right \mathcal{N} -equivariant we have $\|(P_{\mathcal{H}_{\mathcal{N}}}^{\circ} \otimes P_{\mathcal{H}_{\mathcal{N}}}^{\circ})\hat{\alpha}\| \geq \text{ef}(\mathcal{N}, \phi|_{\mathcal{N}})|\langle (E\alpha)^{\wedge}, \xi \otimes \xi \rangle|$, where of course $P_{\mathcal{H}_{\mathcal{N}}}^{\circ}$ is the orthogonal projection of \mathcal{H} onto $\mathcal{H}_{\mathcal{N}} = \mathcal{H}_{\mathcal{N}} \ominus \mathbf{C}\xi$. But $\mathcal{H}_{\mathcal{N}} = \mathcal{H}_{\mathcal{N}} \cap \mathcal{H}$, so $\|(P_{\mathcal{H}_{\mathcal{N}}}^{\circ} \otimes P_{\mathcal{H}_{\mathcal{N}}}^{\circ})\hat{\alpha}\| \geq \|(P_{\mathcal{H}_{\mathcal{N}}}^{\circ} \otimes P_{\mathcal{H}_{\mathcal{N}}}^{\circ})\hat{\alpha}\| \geq \text{ef}(\mathcal{N}, \phi|_{\mathcal{N}})|\langle (E\alpha)^{\wedge}, \xi \otimes \xi \rangle| = \text{ef}(\mathcal{N}, \phi|_{\mathcal{N}})|\langle \hat{\alpha}, \xi \otimes \xi \rangle|$. \square

Theorem 2.5. *Let (\mathcal{M}_i, ϕ_i) , $(i = 1, 2)$ be von Neumann algebras with faithful normal states, and let $(\mathcal{M}, \phi) = (\mathcal{M}_1, \phi_1) * (\mathcal{M}_2, \phi_2)$. Suppose that the linear dimensions satisfy $\dim \mathcal{M}_i \geq 2$, $(i = 1, 2)$ and $\dim \mathcal{M}_1 + \dim \mathcal{M}_2 \geq 5$, and that $\text{ef}(\mathcal{M}_1, \phi_1)\text{ef}(\mathcal{M}_2, \phi_2) \geq 1$, (where $0 \cdot (+\infty) = 0$). Then \mathcal{M} is a factor and*

$$T(\mathcal{M}) = I(\mathcal{M}, \phi) = I(\mathcal{M}_1, \phi_1) \cap I(\mathcal{M}_2, \phi_2). \quad (7)$$

The proof follows from the following proposition.

Proposition 2.6. *Under the hypotheses of the above theorem, if $u \in \mathcal{M}$ is a unitary such that $\phi \circ \text{Ad}_u = \phi$ (i.e. u is in the centralizer, \mathcal{M}_{ϕ}) and $\text{Ad}_u(\mathcal{M}_i) \subseteq \mathcal{M}_i$ $(i = 1, 2)$, then $u \in \mathbf{C}1$.*

Proof. Let us use the notation of §1 with index set $I = \{1, 2\}$. Let $AS = \{s = (s_1, \dots, s_n) \in \bigcup_{m \geq 1} \{1, 2\}^m \mid s_j \neq s_{j+1} \forall 1 \leq j \leq n-1\}$ be the set of all alternating sequences of 1's and 2's. For $s = (s_1, \dots, s_n) \in AS$ let $\mathcal{H}_s^{\circ} = \mathcal{H}_{s_1}^{\circ} \otimes \dots \otimes \mathcal{H}_{s_n}^{\circ} \subseteq \mathcal{H}$ and let $P_{\mathcal{H}_s^{\circ}}$ denote the orthogonal projection of \mathcal{H} onto \mathcal{H}_s° . Suppose that $u \notin \mathbf{C}1$ and consider $\hat{u} \in \mathcal{H}$. We will obtain a contradiction by showing that $\|P_{\mathcal{H}_s^{\circ}}(\hat{u})\|$ as s increases in length are too large for \hat{u} to have bounded norm in \mathcal{H} .

Let $s = (s_1, \dots, s_n) \in AS$ and suppose $s_1 = s_n = k'$ and let $k \in \{1, 2\} \setminus \{k'\}$. Suppose $\gamma \in \mathcal{H}_s^{\circ}$, $\|\gamma\| = 1$. We define

$$\mathcal{H}_k \otimes \gamma \otimes \mathcal{H}_k = \mathbf{C}\gamma \oplus (\mathcal{H}_k^{\circ} \otimes \gamma) \oplus (\gamma \otimes \mathcal{H}_k^{\circ}) \oplus (\mathcal{H}_k^{\circ} \otimes \gamma \otimes \mathcal{H}_k^{\circ}),$$

where $\mathcal{H}_k^{\circ} \otimes \gamma$ is the obvious subspace of $\mathcal{H}_k^{\circ} \otimes \mathcal{H}_s^{\circ} = \mathcal{H}_{(k,s)}^{\circ}$, etcetera. $\mathcal{H}_k \otimes \gamma \otimes \mathcal{H}_k$ is isometrically identified with $\mathcal{H}_k \otimes \mathcal{H}_k$ by identifying

$$\begin{array}{lll} \gamma & \text{with} & \xi_k \otimes \xi_k \\ \mathcal{H}_k^{\circ} \otimes \gamma & \text{with} & \mathcal{H}_k^{\circ} \otimes \xi_k \\ \gamma \otimes \mathcal{H}_k^{\circ} & \text{with} & \xi_k \otimes \mathcal{H}_k^{\circ} \\ \mathcal{H}_k^{\circ} \otimes \gamma \otimes \mathcal{H}_k^{\circ} & \text{with} & \mathcal{H}_k^{\circ} \otimes \mathcal{H}_k^{\circ}. \end{array}$$

Let $Q_\gamma : \mathcal{H} \rightarrow \mathcal{H}_k \otimes \mathcal{H}_k$ be the orthogonal projection from \mathcal{H} onto $\mathcal{H}_k \otimes \gamma \otimes \mathcal{H}_k$ followed by the above identification of $\mathcal{H}_k \otimes \gamma \otimes \mathcal{H}_k$ with $\mathcal{H}_k \otimes \mathcal{H}_k$. One sees immediately that for $a \in \mathcal{M}_k$ and $\zeta \in \mathcal{H}$ one has $Q_\gamma(\pi(a)\zeta) = (\pi(a) \otimes 1)Q_\gamma(\zeta)$.

Lemma 2.7. *For $a \in \mathcal{M}$ and γ as above there is a unique $F_{\gamma,a} \in \mathcal{H}_k \otimes \mathcal{M}_k$ given by*

$$\langle F_{\gamma,a}(\eta)v, w \rangle_{\mathcal{H}_k} = \langle av, Q_\gamma^*(\eta \otimes w) \rangle_{\mathcal{H}} = \langle Q_\gamma(av), \eta \otimes w \rangle_{\mathcal{H}_k \otimes \mathcal{H}_k} \quad \forall \eta \in \overline{\mathcal{H}_k} \text{ and } v, w \in \mathcal{H}_k, \quad (8)$$

identifying $\overline{\mathcal{H}_k}$ and \mathcal{H}_k as sets and viewing $\mathcal{H}_k \subseteq \mathcal{H}$.

Proof. Clearly (8) defines a bounded operator $F_{\gamma,a}(\eta)$ on \mathcal{H}_k for every $\eta \in \overline{\mathcal{H}_k}$. Also for every fixed $v \in \mathcal{H}$, $F_{\gamma,a}(\cdot)v$ is Hilbert–Schmidt and $\|F_{\gamma,a}(\cdot)v\|_{HS} \leq \|av\|$. So it suffices to show that $F_{\gamma,a}(\eta) \in \mathcal{M}_k$ for every $\eta \in \overline{\mathcal{H}_k}$ and every $a \in \mathcal{M}$. Consider a word $b = b_1 b_2 \cdots b_m \in \overset{\circ}{\mathcal{M}}_{t_1} \overset{\circ}{\mathcal{M}}_{t_2} \cdots \overset{\circ}{\mathcal{M}}_{t_m}$, where $t = (t_1, \dots, t_m) \in AS$. Then $F_{\gamma,b}$ is a simple tensor in $\mathcal{H}_k \otimes \mathcal{M}_k$. Indeed,

$$F_{\gamma,b} = \begin{cases} \langle \hat{b}_1 \otimes \cdots \otimes \hat{b}_m, \gamma \rangle & \xi \otimes 1 & \text{if } t = s \\ \langle \hat{b}_2 \otimes \cdots \otimes \hat{b}_m, \gamma \rangle & \hat{b}_1 \otimes 1 & \text{if } t = (k, s) \\ \langle \hat{b}_1 \otimes \cdots \otimes \hat{b}_{m-1}, \gamma \rangle & \xi \otimes b_m & \text{if } t = (s, k) \\ \langle \hat{b}_2 \otimes \cdots \otimes \hat{b}_{m-1}, \gamma \rangle & \hat{b}_1 \otimes b_m & \text{if } t = (k, s, k) \\ 0 & & \text{otherwise.} \end{cases}$$

Hence if a is a linear combination of such words, then $F_{\gamma,a} \in \mathcal{M}_k$. However the $*$ -subalgebra of such linear combinations of words is dense in \mathcal{M} , so an arbitrary $a \in \mathcal{M}$ is the weak operator limit of a net $(a_\lambda)_{\lambda \in \Lambda}$ of them, and as is clear from (8), $F_{\gamma,a}(\eta)$ is then the weak operator limit of $(F_{\gamma,a_\lambda}(\eta))_{\lambda \in \Lambda}$, implying that $F_{\gamma,a} \in \mathcal{M}_k$. This proves Lemma 2.7. \square

Continuing with the proof of Proposition 2.6, let $\tau_\iota = \text{Ad}_u|_{\mathcal{M}_\iota}$ ($\iota \in \{1, 2\}$). Since $\phi|_{\mathcal{M}_\iota} \circ \tau_\iota = \phi|_{\mathcal{M}_\iota}$, there is a corresponding unitary $V_\iota \in B(\mathcal{H}_\iota)$ given by $V_\iota \hat{b} = (\tau_\iota(b))^\wedge \forall b \in \mathcal{M}_\iota$. Note that $V_\iota \xi_\iota = \xi_\iota$. One easily checks that $\pi_\iota(\tau_\iota(a)) = V_\iota \pi_\iota(a) V_\iota^* \forall a \in \mathcal{M}_\iota$.

Lemma 2.8. *Let s and γ be as above. Let $G_\gamma = \tau_k \circ F_{\gamma,u} : \overline{\mathcal{H}_k} \rightarrow \mathcal{M}_k$. Then $G_\gamma \in \mathcal{H}_k \otimes \mathcal{M}_k$ and G_γ is left–right \mathcal{M}_k –equivariant.*

Proof. Clearly G_γ is bounded. Note that $\|G_\gamma(\eta)v\| = \|V_k F_{\gamma,u}(\eta) V_k^* v\|$, so $\|G_\gamma(\cdot)v\|_{HS} = \|F_{\gamma,u}(\cdot) V_k^* v\|_{HS}$ and hence $G_\gamma \in \mathcal{H}_k \otimes \mathcal{M}_k$ and $\|G_\gamma\|_{HS} = \|F_{\gamma,u}\|_{HS}$. To show that G_γ is left–right \mathcal{M}_k –equivariant, for $a \in \mathcal{M}_k$, $\eta \in \overline{\mathcal{H}_k}$ and $v, w \in \mathcal{H}_k$ we have

$$\begin{aligned} \langle G_\gamma(\pi(a)\eta)v, w \rangle &= \langle F_{\gamma,u}(\pi(a)\eta) V_k^* v, V_k^* w \rangle = \langle u(V_k^* v), Q_\gamma^*(\pi(a^*)\eta \otimes (V_k^* w)) \rangle = \langle u(V_k^* v), \pi(a^*) Q_\gamma^*(\eta \otimes V_k^* w) \rangle = \\ &= \langle au(V_k^* v), Q_\gamma^*(\eta \otimes V_k^* w) \rangle = \langle (u\tau_k^{-1}(a))(V_k^* v), Q_\gamma^*(\eta \otimes V_k^* w) \rangle = \langle uV_k^*(av), Q_\gamma^*(\eta \otimes V_k^* w) \rangle = \\ &= \langle F_{\gamma,u}(\eta) V_k^*(av), V_k^* w \rangle = \langle G_\gamma(\eta)av, w \rangle, \end{aligned}$$

so $G_\gamma(\pi(a)\eta) = G_\gamma(\eta)a$, i.e. G_γ is left–right \mathcal{M}_k –equivariant. This proves Lemma 2.8. \square

Lemma 2.9. *There is $s = (s_1, \dots, s_n) \in AS$ with $s_1 = s_n$ such that $P_{\mathcal{H}_s}^\circ(\hat{u}) \neq 0$.*

Proof. Since we supposed that $u \notin \mathbf{C}1$, there is $t = (t_1, \dots, t_n) \in AS$ such that $P_{\mathcal{H}_t}^\circ(\hat{u}) \neq 0$. If $t_1 = t_n$ we are done. If $t_1 \neq t_n$ let $k = t_1$, $k' = t_n$. We may suppose $\dim \mathcal{M}_k \geq 3$. (If instead $\dim \mathcal{M}_{k'} \geq 3$ the same argument but reflected will work.) Then there is $\gamma \in \overset{\circ}{\mathcal{H}}_{t'}$, where $t' = (t_2, \dots, t_n)$, such that

$$Q_\gamma(\hat{u}) = r(\xi_k \otimes \xi_k) + \zeta_1 \otimes \xi_k + \xi_k \otimes \zeta_2 + X,$$

where $r \in \mathbf{C}$, $\zeta_1, \zeta_2 \in \overset{\circ}{\mathcal{H}}_k$, $X \in \overset{\circ}{\mathcal{H}}_k \otimes \overset{\circ}{\mathcal{H}}_k$ and $\zeta_1 \neq 0$. To prove the lemma it will suffice to show that $X \neq 0$, because then $P_{\overset{\circ}{\mathcal{H}}_{(t,k)}}(\hat{u}) \neq 0$. Suppose for contradiction that $X = 0$. Then

$$\langle G_\gamma(\eta)\xi_k, w \rangle = \langle Q_\gamma(\hat{u}), \eta \otimes V_k^* w \rangle = r\langle \xi_k, \eta \rangle \langle \xi_k, w \rangle + \langle \zeta_1, \eta \rangle \langle \xi_k, w \rangle + \langle \xi_k, \eta \rangle \langle V_k \zeta_2, w \rangle,$$

so $G_\gamma(\eta)\xi_k = \langle r\xi_k + \zeta_1, \eta \rangle \xi_k + \langle \xi_k, \eta \rangle V_k \zeta_2 \in (\mathcal{M}_k)^\wedge \forall \eta \in \overline{\mathcal{H}}_k$. Hence $V_k \zeta_2 = \hat{b}$, some $b \in \mathcal{M}_k$ and

$$G_\gamma = (r\xi_k + \zeta_1) \otimes 1 + \xi_k \otimes b.$$

From the left-right \mathcal{M}_k -equivariance of G_γ we have that

$$a(r\xi_k + \zeta_1) \otimes 1 + \hat{a} \otimes b = (r\xi_k + \zeta_1) \otimes a + \xi_k \otimes ba \quad \forall a \in \mathcal{M}_k. \quad (9)$$

If $b = 0$ then we can get a contradiction to (9) by choosing any $a \notin \mathbf{C}1$, (remembering that $\zeta_1 \neq 0$ and $\zeta_1 \perp \xi_k$). If $b \neq 0$ then since $\dim \mathcal{M}_k \geq 3$ there is $a \in \mathcal{M}_k$ such that $\hat{a} \perp \xi_k$ and $\hat{a} \perp \zeta_1$. For this a , we have that the right hand side of (9) is an element of $\mathcal{H}_k \otimes \mathcal{M}_k$ sending $\hat{a} \in \overline{\mathcal{H}}_k$ to zero while the left hand side sends \hat{a} to some multiple of 1 plus $\langle \hat{a}, \hat{a} \rangle b$, which is nonzero, a contradiction. Hence X must be nonzero and Lemma 2.9 is proved. \square

Lemma 2.10. *Let $s = (s_1, \dots, s_n) \in AS$ with $s_1 = s_n = k'$. Let $k \in \{1, 2\} \setminus \{k'\}$ and $t = (k, s_1, s_2, \dots, s_n, k)$. Then $\|P_{\overset{\circ}{\mathcal{H}}_t}(\hat{u})\| \geq \text{ef}(\mathcal{M}_k, \phi_k) \|P_{\overset{\circ}{\mathcal{H}}_s}(\hat{u})\|$.*

Proof. Let $\{\gamma_j\}_{j \in J}$ be an orthonormal basis for $\overset{\circ}{\mathcal{H}}_s$. Then

$$\begin{aligned} \|P_{\overset{\circ}{\mathcal{H}}_s}(\hat{u})\|^2 &= \sum_{j \in J} |\langle \gamma_j, \hat{u} \rangle|^2 = \sum_{j \in J} |\langle \xi_k \otimes \xi_k, Q_\gamma(\hat{u}) \rangle|^2 \\ \|P_{\overset{\circ}{\mathcal{H}}_t}(\hat{u})\|^2 &= \sum_{j \in J} \|(P_{\overset{\circ}{\mathcal{H}}_k} \otimes P_{\overset{\circ}{\mathcal{H}}_k})Q_\gamma(\hat{u})\|^2. \end{aligned} \quad (10)$$

But note that for $\gamma \in \{\gamma_j\}_{j \in J}$,

$$\langle Q_\gamma(\hat{u}), \xi_k \otimes \xi_k \rangle = \langle u(V_k^* \xi_k), Q_\gamma^*(\xi_k \otimes V_k^* \xi_k) \rangle = \langle G_\gamma(\xi_k) \xi_k, \xi_k \rangle = \langle \hat{G}_\gamma, \xi_k \otimes \xi_k \rangle,$$

so $\|P_{\overset{\circ}{\mathcal{H}}_s}(\hat{u})\|^2 = \sum_{j \in J} |\langle \hat{G}_\gamma, \xi_k \otimes \xi_k \rangle|$. Let $\{v_\lambda\}_{\lambda \in \Lambda}$ be an orthonormal basis for $\overset{\circ}{\mathcal{H}}_k$. Then for $\gamma \in \{\gamma_j\}_{j \in J}$,

$$\begin{aligned} \|(P_{\overset{\circ}{\mathcal{H}}_k} \otimes P_{\overset{\circ}{\mathcal{H}}_k})Q_\gamma(\hat{u})\|^2 &= \sum_{\lambda, \mu \in \Lambda} |\langle u(V_k^* \xi_k), Q_\gamma^*(v_\lambda \otimes V_k^* v_\mu) \rangle|^2 = \sum_{\lambda, \mu \in \Lambda} |\langle G_\gamma(v_\lambda) \xi_k, v_\mu \rangle|^2 \\ &= \sum_{\lambda, \mu \in \Lambda} |\langle \hat{G}_\gamma, v_\lambda \otimes v_\mu \rangle|^2 = \|(P_{\overset{\circ}{\mathcal{H}}_k} \otimes P_{\overset{\circ}{\mathcal{H}}_k})(\hat{G}_\gamma)\|^2. \end{aligned}$$

But since G_γ is left-right \mathcal{M}_k -equivariant, we have by definition that

$$\|(P_{\overset{\circ}{\mathcal{H}}_k} \otimes P_{\overset{\circ}{\mathcal{H}}_k})(\hat{G}_\gamma)\|^2 \geq \text{ef}(\mathcal{M}_k, \phi_k)^2 |\langle \hat{G}_\gamma, \xi_k \otimes \xi_k \rangle|^2.$$

Now use this together with (10) to prove Lemma 2.10. \square

Proof of Proposition 2.6. Lemma 2.9 guarantees us that there is $s = (s_1, \dots, s_n) \in AS$ such that $s_1 = s_n = k'$ and $P_{\overset{\circ}{\mathcal{H}}_s}(\hat{u}) \neq 0$. Let $k \in \{1, 2\} \setminus \{k'\}$ and

$$t(m) = (\underbrace{k', k, \dots, k', k}_{m \text{ times } k', k}, s_1, \dots, s_n, \underbrace{k, k', \dots, k, k'}_{m \text{ times } k, k'}).$$

Then by Lemma 2.10,

$$\|P_{\mathcal{H}_{t(m)}}^o(\hat{u})\|^2 \leq \text{ef}(\mathcal{M}_{k'}, \phi_{k'})^{2m} \text{ef}(\mathcal{M}_k, \phi_k)^{2m} \|P_{\mathcal{H}_s}^o(\hat{u})\|^2.$$

Under the hypotheses of Theorem 2.5, this gives a contradiction to $\hat{u} \in \mathcal{H}$. \square

Proof of Theorem 2.5. To prove the factoriality of \mathcal{M} it suffices to show that any unitary u that is in the center of \mathcal{M} must be a constant. However, such a unitary satisfies the hypotheses of Proposition 2.6, so is constant. To prove (7), suppose $t \in T(\mathcal{M})$, i.e. $\sigma_t = \text{Ad}_u$, $u \in \mathcal{M}$. Of course σ_t preserves ϕ , and by Theorem 1 we know that σ_t sends \mathcal{M}_ι into \mathcal{M}_ι ($\iota = 1, 2$), so again by Proposition 2.6 we must have that u is constant, so that $t \in I(\mathcal{M}, \phi)$. Thus $T(\mathcal{M}) = I(\mathcal{M}, \phi)$. But also by Theorem 1 we have the last equality in (7). \square

§3 Calculations.

In this section we calculate the expansion factors of various algebras. Actually, the methods used here allow one, with a bit of work, to calculate $\text{ef}(\mathcal{M}, \phi)$ for any pair (\mathcal{M}, ϕ) . But since Theorem 2.5 is sufficient but not necessary for factoriality, (see Remark 4.1), It would not be rewarding to compute $\text{ef}(\mathcal{M}, \phi)$ for the most general (\mathcal{M}, ϕ) . We compute enough special cases to show (Lemma 3.2) that the hypotheses of Theorem 2.5 are satisfied in gratifyingly many cases, and also to illustrate some of the limitations of Theorem 2.5, (see Remark 4.1).

Lemma 3.1. *Let A be a finite dimensional commutative algebra with faithful state ϕ . Suppose e_1, \dots, e_n are the minimal projections of A having traces $\phi(e_\iota) = \lambda_\iota$, ($1 \leq \iota \leq n$), with $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n > 0$. Then*

$$\text{ef}(A, \phi)^2 = 1 + \left(\sum_{\iota=1}^n \frac{\lambda_\iota^2}{1 - 2\lambda_\iota} \right)^{-1}, \quad (11)$$

where if $\lambda_1 = \frac{1}{2}$ we take the right hand side of (11) to be 1.

Proof. We will use the method described in Remark 2.2 to calculate the expansion factor. $\mathcal{H} = L^2(A, \phi)$ has orthogonal basis $\{\hat{e}_\iota \mid 1 \leq \iota \leq n\}$. Hence every $X \in \mathcal{H} \otimes \mathcal{H}$ can be written $X = \sum_{\iota, j=1}^n c_{\iota, j} \hat{e}_\iota \otimes \hat{e}_j$. Since for X to be left-right A -invariant means $(\pi(e_p) \otimes 1)X = (1 \otimes \rho(e_p))X$, we see that $c_{\iota, j} = 0$ if $\iota \neq j$ and that X is left-right A -invariant if and only if it is of the form

$$X = \sum_{\iota=1}^n c_\iota \hat{e}_\iota \otimes \hat{e}_\iota,$$

for $c_\iota \in \mathbb{C}$. Now in \mathcal{H} , $\|\hat{e}_\iota\|^2 = \phi(e_\iota e_\iota) = \lambda_\iota$ and $P_\xi(\hat{e}_\iota) = \lambda_\iota \xi$, so

$$\begin{aligned} (P_\xi \otimes P_\xi)(X) &= \left(\sum_{\iota=1}^n c_\iota \lambda_\iota^2 \right) \xi \otimes \xi, & \|(P_\xi \otimes P_\xi)(X)\|^2 &= \left| \sum_{\iota=1}^n c_\iota \lambda_\iota^2 \right|^2, \\ (P_\xi \otimes I)(X) &= \sum_{\iota=1}^n c_\iota \lambda_\iota (\xi \otimes \hat{e}_\iota), & \|(P_\xi \otimes I)(X)\|^2 &= \sum_{\iota=1}^n |c_\iota|^2 \lambda_\iota^3, \\ (I \otimes P_\xi)(X) &= \sum_{\iota=1}^n c_\iota \lambda_\iota (\hat{e}_\iota \otimes \xi), & \|(I \otimes P_\xi)(X)\|^2 &= \sum_{\iota=1}^n |c_\iota|^2 \lambda_\iota^3, \\ & & \|X\|^2 &= \sum_{\iota=1}^n |c_\iota|^2 \lambda_\iota^2. \end{aligned}$$

Thus

$$\begin{aligned} \|(P_{\mathcal{H}}^o \otimes P_{\mathcal{H}}^o)(X)\|^2 &= \|X\|^2 - \|(P_\xi \otimes I)(X)\|^2 - \|(I \otimes P_\xi)(X)\|^2 + \|(P_\xi \otimes P_\xi)(X)\|^2 \\ &= \sum_{\iota=1}^n |c_\iota|^2 (\lambda_\iota^2 - 2\lambda_\iota^3) + \left| \sum_{\iota=1}^n c_\iota \lambda_\iota^2 \right|^2. \end{aligned} \quad (12)$$

As remarked in 2.2, $\text{ef}(A, \phi)^2$ is the infimum of (12) as $(c_\iota)_{1 \leq \iota \leq n}$ varies over \mathbf{C}^n , subject to the constraint

$$C = \sum_{\iota=1}^n c_\iota \lambda_\iota^2 = 1, \quad (13)$$

i.e. the infimum of

$$V = 1 + \sum_{\iota=1}^n |c_\iota|^2 \lambda_\iota^2 (1 - 2\lambda_\iota) \quad (14)$$

subject to (13).

When $\lambda_1 \leq \frac{1}{2}$, clearly $V \geq 1$ everywhere, and for the special case $\lambda_1 = \frac{1}{2}$ we can chose $c_1 = 4$ and $c_\iota = 0$ for $2 \leq \iota \leq n$ to give $V = 1$, so the infimum equals 1, as required. We may thus assume henceforth that $\lambda_1 \neq \frac{1}{2}$.

Let us show that it suffices to find the infimum of V subject to (13) for $(c_\iota)_{1 \leq \iota \leq n} \in \mathbf{R}^n$. Letting $c_\iota = a_\iota + ib_\iota$ for $a_\iota, b_\iota \in \mathbf{R}$, it suffices to show that $V' = \sum_{\iota=1}^n b_\iota^2 \lambda_\iota^2 (1 - 2\lambda_\iota)$ is always non-negative when $C' = \sum_{\iota=1}^n b_\iota \lambda_\iota^2 = 0$. But if some such choice of $(b_\iota)_{1 \leq \iota \leq n}$ gives negative V' , then multiples of them also satisfy $C' = 0$ and give V' as large and negative as we please, enough to make V itself negative. This is a contradiction, since V , being the square-norm of a vector, must be nonnegative. Hence we may assume $c_\iota \in \mathbf{R} \forall \iota$.

In order to use the method of Lagrange multipliers, we need to know that the infimum of V subject to (13) occurs at a relative minimum of V on the manifold defined by (13). For $\lambda_1 < \frac{1}{2}$, this is obvious, since it is then clear that $V \rightarrow +\infty$ as $(c_\iota)_{1 \leq \iota \leq n} \rightarrow \infty$. For $\lambda_1 > \frac{1}{2}$ we proceed as follows. For $s \in \mathbf{R}$ let $V(s)$ denote the minimum value of V subject to (13) with the value of c_1 fixed to be s . We will show that $V(s) \rightarrow +\infty$ as $s \rightarrow \pm\infty$. It is clear that with c_1 fixed, $V \rightarrow +\infty$ as $(c_\iota)_{2 \leq \iota \leq n} \rightarrow \infty$, so we may use the method of Lagrange multipliers to find $V(s)$. We have

$$\frac{\partial C}{\partial c_\iota} = \lambda_\iota^2 \quad (15)$$

and

$$\frac{\partial V}{\partial c_\iota} = 2c_\iota \lambda_\iota^2 (1 - 2\lambda_\iota), \quad (16)$$

so the value of $V(s)$ occurs where

$$c_\iota = \frac{r}{2(1 - 2\lambda_\iota)}, \quad (2 \leq \iota \leq n),$$

for some $r \in \mathbf{R}$. From (13) we have that

$$r = (1 - s\lambda_1^2) \left(\sum_{\iota=2}^n \frac{\lambda_\iota^2}{2(1 - 2\lambda_\iota)} \right)^{-1},$$

so

$$\begin{aligned} V(s) &= 1 + s^2 \lambda_1^2 (1 - 2\lambda_1) + r^2 \sum_{\iota=2}^n \frac{\lambda_\iota^2 (1 - 2\lambda_\iota)}{4(1 - 2\lambda_\iota)^2} \\ &= 1 + s^2 \lambda_1^2 (1 - 2\lambda_1) + \frac{1}{2} (1 - s\lambda_1^2)^2 \left(\sum_{\iota=2}^n \frac{\lambda_\iota^2}{2(1 - 2\lambda_\iota)} \right)^{-1} \\ &= s^2 \lambda_1^2 (1 - 2\lambda_1) + \frac{1}{2} s^2 \lambda_1^4 \left(\sum_{\iota=2}^n \frac{\lambda_\iota^2}{2(1 - 2\lambda_\iota)} \right)^{-1} + \text{lower terms in } s. \end{aligned}$$

We must show that

$$\frac{1}{2} \lambda_1^4 \left(\sum_{\iota=2}^n \frac{\lambda_\iota^2}{2(1 - 2\lambda_\iota)} \right)^{-1} > \lambda_1^2 (2\lambda_1 - 1),$$

i.e.

$$\frac{\lambda_1^2}{2\lambda_1 - 1} > \sum_{\iota=2}^n \frac{\lambda_\iota^2}{1 - 2\lambda_\iota}. \quad (17)$$

Consider the function $f(t) = \frac{t^2}{1-2t}$. One easily verifies that $f(0) = 0$ and that on $(0, \frac{1}{2})$, $f'(t)$ and $f''(t)$ are strictly positive. Thus if $t_1, t_2 \in (0, \frac{1}{2})$, $t_1 \leq t_2$ and $t_1 + t_2 < \frac{1}{2}$, then the point $(t_1 + t_2, f(t_1 + t_2))$ lies above the line $y = \frac{f(t_2)}{t_2}x$, and the point $(t_1, f(t_1))$ lies below this line. Hence

$$f(t_1) + f(t_2) \leq \frac{f(t_2)}{t_2}t_1 + f(t_2) \leq f(t_1 + t_2).$$

Consequently we have

$$\sum_{\iota=2}^n \frac{\lambda_\iota^2}{1 - 2\lambda_\iota} = \sum_{\iota=2}^n f(\lambda_\iota) \leq f\left(\sum_{\iota=2}^n \lambda_\iota\right) = f(1 - \lambda_1) = \frac{(1 - \lambda_1)^2}{2\lambda_1 - 1}.$$

Thus to show (17) it suffices to show

$$\frac{\lambda_1^2}{2\lambda_1 - 1} > \frac{(1 - \lambda_1)^2}{2\lambda_1 - 1},$$

but since $\lambda_1 > \frac{1}{2}$, this is true. We have thus shown that $V(s)$ is a quadratic polynomial in s with positive leading coefficient, so $V(s) \rightarrow +\infty$ as $s \rightarrow \pm\infty$.

We are now justified in using the method of Lagrange multipliers to find the minimum of (14) subject to the constraint (13). From (15) and (16) we see that the minimum value of V occurs when

$$c_\iota = \frac{r}{2(1 - 2\lambda_\iota)}, \quad (1 \leq \iota \leq n)$$

for some $r \in \mathbf{R}$. From (13) we get

$$r = \left(\sum_{\iota=1}^n \frac{\lambda_\iota^2}{2(1 - 2\lambda_\iota)} \right)^{-1},$$

so the value of V at the global minimum is

$$V_{\min} = 1 + r^2 \sum_{\iota=1}^n \frac{\lambda_\iota^2(1 - 2\lambda_\iota)}{4(1 - 2\lambda_\iota)^2} = 1 + \frac{r}{2} = 1 + \left(\sum_{\iota=1}^n \frac{\lambda_\iota^2}{1 - 2\lambda_\iota} \right)^{-1}.$$

□

Lemma 3.2. *Let \mathcal{M} be a von Neumann algebra and ϕ a faithful state on \mathcal{M} . If there are no minimal projections $e \in \mathcal{M}$ such that $\phi(e) > \frac{1}{2}$, then $\text{ef}(\mathcal{M}, \phi) \geq 1$.*

Proof. We can write $\mathcal{M} = \mathcal{A}_1 \oplus \mathcal{A}_2$, where \mathcal{A}_2 is diffuse and where \mathcal{A}_1 has the property that every central projection contains a nontrivial minimal projection. (Thus \mathcal{A}_1 is a possibly infinite direct sum of type I factors.) Let $\mathcal{H}_\iota = L^2(\mathcal{A}_\iota, \phi|_{\mathcal{A}_\iota})$ ($\iota = 1, 2$), $\mathcal{H} = L^2(\mathcal{M}, \phi) = \mathcal{H}_1 \oplus \mathcal{H}_2$ and consider $\alpha \in \mathcal{H} \otimes \mathcal{M}$ that is left-right \mathcal{M} -equivariant. Let $p \in \mathcal{M}$ be the central projection $1 \oplus 0$. Then since $\alpha(\pi(p)\eta) = \alpha(\eta)p \ \forall \eta \in \overline{\mathcal{H}} = \overline{\mathcal{H}}_1 \oplus \overline{\mathcal{H}}_2$, we have that $\alpha = \alpha_1 \oplus \alpha_2$, where $\alpha_\iota : \mathcal{H}_\iota \rightarrow \mathcal{A}_\iota$ is left-right \mathcal{A}_ι -equivariant. But $\alpha_2 = 0$ by Lemma 2.3. Thus (\mathcal{M}, ϕ) has the same expansion factor as what we get when we replace \mathcal{A}_2 by a diffuse commutative von Neumann algebra, so we assume without loss of generality that \mathcal{A}_2 is commutative. Then there is a commutative subalgebra $\mathcal{B} \subseteq \mathcal{M}$ having no minimal projections $e \in \mathcal{B}$ with $\phi(e) > \frac{1}{2}$ and a conditional expectation $E : \mathcal{M} \rightarrow \mathcal{B}$ such that $\phi \circ E = \phi$. Indeed, since \mathcal{A}_1 is a direct sum of type I factors, we need only see that a type I factor \mathcal{N} with faithful state ψ has a commutative subalgebra $\mathcal{C} \in \mathcal{N}$ and a ψ -preserving conditional expectation of \mathcal{N} onto \mathcal{C} . However, $\psi(\cdot) = \text{Tr}(h\cdot)$, where Tr is a faithful trace on \mathcal{N} and where $h \in \mathcal{N}$ is a positive trace-class operator, each of whose spectral projections is a finite projection. Let \mathcal{C}

be the MASA containing all the spectral projections of h . Let $(e_\iota)_{\iota \in I}$ be the set of minimal projections of \mathcal{C} and let $E : \mathcal{N} \rightarrow \mathcal{C}$ be $E(x) = \sum_{\iota \in I} \psi(e_\iota x e_\iota) e_\iota$. Then E is a ψ -preserving conditional expectation. Thus we have shown the existence of commutative \mathcal{B} as described above and a ϕ -preserving conditional expectation. So by Lemma 2.4, $\text{ef}(\mathcal{M}, \phi) \geq \text{ef}(\mathcal{B}, \phi|_{\mathcal{B}})$. But \mathcal{B} has a finite dimensional subalgebra \mathcal{B}_0 having no minimal projections e with $\phi(e) > \frac{1}{2}$, and \mathcal{B}_0 is of course the image of a ϕ -preserving conditional expectation $E_0 : \mathcal{B} \rightarrow \mathcal{B}_0$, so applying Lemma 2.4 again yields $\text{ef}(\mathcal{M}, \phi) \geq \text{ef}(\mathcal{B}_0, \phi|_{\mathcal{B}_0})$. An application of Lemma 3.1 now completes the proof. \square

Lemma 3.3. *Let $A = M_n(\mathbf{C})$ be the algebra of $n \times n$ complex matrices, containing a system of matrix units $\{e_{\iota j} \mid 1 \leq \iota, j \leq n\}$ and equipped with a faithful state ϕ such that*

$$\phi(e_{\iota j}) = \begin{cases} \lambda_\iota & \text{if } \iota = j \\ 0 & \text{if } \iota \neq j, \end{cases}$$

where $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n > 0$. Then

$$\text{ef}(A, \phi)^2 = 1 + \left(\sum_{p=1}^n \frac{\lambda_p^3}{1 - 2\lambda_p^2} \right)^{-1},$$

where we interpret the right hand side of the above equation to be 1 if $\lambda_1 = \frac{1}{\sqrt{2}}$.

Proof. The proof goes pretty much exactly as that of Lemma 3.1. The set $\{\hat{e}_{\iota j} \mid 1 \leq \iota, j \leq n\}$ is an orthogonal basis for $\mathcal{H} = L^2(A, \phi)$, and an arbitrary element $X \in \mathcal{H} \otimes \mathcal{H}$ is $X = \sum_{\iota, j, k, l} c_{\iota j k l} \hat{e}_{\iota j} \otimes \hat{e}_{k l}$. For X to be left-right A -equivariant means $(\pi(e_{pq}) \otimes 1)X = (1 \otimes \rho(e_{pq}))X$, i.e.

$$\sum_{1 \leq j, k, l \leq n} c_{q, j, k, l} \hat{e}_{pj} \otimes \hat{e}_{kl} = \sum_{1 \leq \iota, j, k \leq n} c_{\iota, j, k, p} \hat{e}_{\iota j} \otimes \hat{e}_{kp}.$$

Thus $c_{\iota, j, k, l} = 0$ unless $\iota = l$ and $c_{p, j, k, p} = c_{q, j, k, q} \forall 1 \leq p, q \leq n$, so $X \in \mathcal{H} \otimes \mathcal{H}$ is left-right equivariant if and only if it is of the form

$$X = \sum_{j, k=1}^n c_{j, k} \left(\sum_{p=1}^n \hat{e}_{pj} \otimes \hat{e}_{kp} \right)$$

for $c_{j, k} \in \mathbf{C}$. In \mathcal{H} we have $\|\hat{e}_{st}\|^2 = \phi(e_{st}^* e_{st}) = \phi(e_{tt}) = \lambda_t$ and $P_\xi(\hat{e}_{pq}) = \delta_{pq} \lambda_p \xi$, so we get

$$\begin{aligned} (P_\xi \otimes P_\xi)(X) &= \left(\sum_{p=1}^n c_{p, p} \lambda_p^2 \right) \xi \otimes \xi, & \|(P_\xi \otimes P_\xi)(X)\|^2 &= \left| \sum_{p=1}^n c_{p, p} \lambda_p^2 \right|^2, \\ (P_\xi \otimes I)(X) &= \sum_{j, k=1}^n c_{j, k} \lambda_j (\xi \otimes \hat{e}_{kj}), & \|(P_\xi \otimes I)(X)\|^2 &= \sum_{j, k=1}^n |c_{j, k}|^2 \lambda_j^3, \\ (I \otimes P_\xi)(X) &= \sum_{j, k=1}^n c_{j, k} \lambda_k (\hat{e}_{kj} \otimes \xi), & \|(I \otimes P_\xi)(X)\|^2 &= \sum_{j, k=1}^n |c_{j, k}|^2 \lambda_j \lambda_k^2, \end{aligned}$$

$$\|X\|^2 = \sum_{j, k=1}^n |c_{j, k}|^2 \sum_{p=1}^n \lambda_j \lambda_p = \sum_{j, k=1}^n |c_{j, k}|^2 \lambda_j,$$

$$\begin{aligned} \|(P_{\mathcal{H}} \otimes P_{\mathcal{H}})(X)\|^2 &= \|X\|^2 - \|(P_\xi \otimes I)(X)\|^2 - \|(I \otimes P_\xi)(X)\|^2 + \|(P_\xi \otimes P_\xi)(X)\|^2 \\ &= \sum_{j, k=1}^n |c_{j, k}|^2 \lambda_j (1 - \lambda_j^2 - \lambda_k^2) + \left| \sum_{p=1}^n c_{p, p} \lambda_p^2 \right|^2. \end{aligned}$$

Now $\text{ef}(A, \phi)^2$ will equal the infimum of $V = 1 + \sum_{j,k=1}^n |c_{j,k}|^2 \lambda_j (1 - \lambda_j^2 - \lambda_k^2)$ as $(c_{j,k})_{1 \leq j,k \leq n}$ ranges over \mathbf{C}^{n^2} subject to the constraint $C = \sum_{p=1}^n c_p \lambda_p^2 = 1$. Note that if $j \neq k$ then since $\lambda_j + \lambda_k \leq 1$ we have $\lambda_j^2 + \lambda_k^2 < 1$. Hence we may assume without loss of generality that $c_{j,k} = 0$ if $j \neq k$, and rewriting $c_{p,p} = c_p$, we want to find the infimum of

$$V = 1 + \sum_{p=1}^n |c_p|^2 \lambda_p (1 - 2\lambda_p^2) \quad (18)$$

as $(c_p)_{1 \leq p \leq n}$ ranges over \mathbf{C}^n subject to the constraint

$$C = \sum_{p=1}^n c_p \lambda_p^2 = 1. \quad (19)$$

When $\lambda_1 \leq \frac{1}{\sqrt{2}}$, clearly $V \geq 1$ everywhere. For the special case $\lambda_1 = \frac{1}{\sqrt{2}}$ we get $V = 1$ by setting $c_1 = 2$ and $c_p = 0$ for $2 \leq p \leq n$. Thus the infimum is 1, as required, and we may henceforth assume that $\lambda_1 \neq \frac{1}{\sqrt{2}}$.

We can easily show that we may restrict ourselves to considering $(c_p)_{1 \leq p \leq n} \in \mathbf{R}^n$, exactly as we did in the proof of Lemma 3.1.

Also as in the proof of Lemma 3.1, the infimum of V on the manifold defined by (19) clearly occurs at a local minimum if $\lambda_1 < \frac{1}{\sqrt{2}}$, but we need to argue further to show this if $\lambda_1 > \frac{1}{\sqrt{2}}$. Again let $V(s)$ for $s \in \mathbf{R}$ be the minimum of V on the manifold defined by (19) when c_1 is fixed to have the value s . We may clearly use the method of Lagrange multipliers to find $V(s)$. We have

$$\frac{\partial C}{\partial c_p} = \lambda_p^2 \quad (20)$$

and

$$\frac{\partial V}{\partial c_p} = 2c_p \lambda_p (1 - 2\lambda_p^2), \quad (21)$$

so the value of $V(s)$ occurs where

$$c_p = \frac{r \lambda_p}{2(1 - 2\lambda_p^2)}, \quad (2 \leq p \leq n),$$

for some $r \in \mathbf{R}$. From (19) we obtain

$$r = (1 - s\lambda_1^2) \left(\sum_{p=2}^n \frac{\lambda_p^3}{2(1 - 2\lambda_p^2)} \right)^{-1}.$$

Thus

$$\begin{aligned} V(s) &= 1 + s^2 \lambda_1 (1 - 2\lambda_1^2) + r^2 \sum_{p=2}^n \frac{\lambda_p^3 (1 - 2\lambda_p^2)}{4(1 - 2\lambda_p^2)^2} \\ &= 1 + s^2 \lambda_1 (1 - 2\lambda_1^2) + \frac{1}{2} r^2 \sum_{p=2}^n \frac{\lambda_p^3}{2(1 - 2\lambda_p^2)} \\ &= 1 + s^2 \lambda_1 (1 - 2\lambda_1^2) + \frac{1}{2} (1 - s\lambda_1^2)^2 \left(\sum_{p=2}^n \frac{\lambda_p^3}{2(1 - 2\lambda_p^2)} \right)^{-1}. \end{aligned} \quad (22)$$

Now (22) is a quadratic polynomial in s , and the coefficient of s^2 is

$$\lambda_1 (1 - 2\lambda_1^2) + \frac{\lambda_1^4}{2} \left(\sum_{p=2}^n \frac{\lambda_p^3}{2(1 - 2\lambda_p^2)} \right)^{-1},$$

which we want to show is positive, *i.e.*

$$\frac{\lambda_1^3}{2} > (2\lambda_1^2 - 1) \left(\sum_{p=2}^n \frac{\lambda_p^3}{2(1 - 2\lambda_p^2)} \right),$$

i.e.

$$\frac{\lambda_1^3}{2\lambda_1^2 - 1} > \sum_{p=2}^n \frac{\lambda_p^3}{1 - 2\lambda_p^2}.$$

Consider the functions $g(t) = \frac{t^3}{1-2t^2}$. Just as we did in the proof of Lemma 3.1 for f , we can easily show that $g(t_1) + g(t_2) \leq g(t_1 + t_2)$ when t_1, t_2 and $t_1 + t_2$ are in $(0, \frac{1}{2})$. But then we have

$$\sum_{p=2}^n \frac{\lambda_p^3}{1 - 2\lambda_p^2} = \sum_{p=2}^n g(\lambda_p) \leq g\left(\sum_{p=2}^n \lambda_p\right) = g(1 - \lambda_1) = \frac{(1 - \lambda_1)^3}{1 - 2(1 - \lambda_1)^2}.$$

So we would like to show that

$$\frac{\lambda_1^3}{2\lambda_1^2 - 1} > \frac{(1 - \lambda_1)^3}{1 - 2(1 - \lambda_1)^2} \text{ for } \frac{1}{\sqrt{2}} < \lambda_1 < 1,$$

which works out to showing that the polynomial $p(\lambda_1) = -2\lambda_1^4 + 4\lambda_1^3 + \lambda_1^2 - 3\lambda_1 + 1 > 0$ for $\frac{1}{\sqrt{2}} < \lambda_1 < 1$. One finds that the roots of $p(t) = 0$ are $t = \frac{1 \pm \sqrt{4 \pm \sqrt{17}}}{2}$, and thus the real roots are approximately -0.925 and 1.925 . Since $p(t) \rightarrow -\infty$ as $t \rightarrow \pm\infty$, this shows that $p(\lambda_1) > 0$ for $\frac{1}{\sqrt{2}} < \lambda_1 < 1$. Thus we are justified in using the method of Lagrange multipliers to find V .

From (20) and (21) we have that

$$c_p = \frac{r\lambda_p}{2(1 - 2\lambda_p^2)} \quad (1 \leq p \leq n)$$

for some $r \in \mathbf{R}$, and hence from (19) we get

$$r = \left(\sum_{p=1}^n \frac{\lambda_p^3}{2(1 - 2\lambda_p^2)} \right)^{-1}.$$

Thus from (18), the value of V at its global minimum is

$$V_{\min} = 1 + r^2 \sum_{p=1}^n \frac{\lambda_p^3}{4(1 - 2\lambda_p^2)} = 1 + \frac{r}{2} = 1 + \left(\sum_{p=1}^n \frac{\lambda_p^3}{1 - 2\lambda_p^2} \right)^{-1}.$$

□

Examples 3.4.

i) If $A = \mathbf{C}^n$ ($n \geq 2$) and ϕ is the state giving each minimal projection weight $\frac{1}{n}$, then

$$\text{ef}(A, \phi)^2 = 1 + \left(n \frac{\frac{1}{n^2}}{1 - \frac{2}{n}} \right)^{-1} = n - 1.$$

ii) If $A = M_n(\mathbf{C})$ with τ the normalized trace, then

$$\text{ef}(A, \tau)^2 = 1 + \left(n \frac{\frac{1}{n^3}}{1 - \frac{2}{n^2}} \right)^{-1} = n^2 - 1.$$

iii) If $A = \mathbf{C} \oplus \mathbf{C}$ and ϕ is the state assigning to one of the minimal projections the value $\lambda > \frac{1}{2}$, then

$$\text{ef}(A, \phi)^2 = 1 + \left(\frac{\lambda^2}{1 - 2\lambda} + \frac{(1 - \lambda)^2}{2\lambda - 1} \right)^{-1} = 0.$$

iv) If $A = M_2(\mathbf{C})$, if ϕ is as described in Lemma 3.3 and if we write $\lambda = \lambda_1 \in [\frac{1}{2}, 1)$, then

$$\text{ef}(A, \phi)^2 = 1 + \left(\frac{\lambda^3}{1 - 2\lambda^2} + \frac{(1 - \lambda)^3}{1 - 2(1 - \lambda)^2} \right)^{-1} = \frac{\lambda + \lambda^2 - 4\lambda^3 + 2\lambda^4}{1 - 3\lambda + \lambda^2 + 4\lambda^3 - 2\lambda^4}.$$

§4. Consequences and limitations of the main theorem.

Remark 4.1. *Factoriality.*

- i) From the calculations in §3, see specifically Lemma 3.2, we see that the conditions of Theorem 2.5 are satisfied for algebras whose minimal projections are not too big. This may include “most” algebras of interest. However, Theorem 2.5 is a far from satisfactory answer to the question about factoriality. Indeed, consider $(\mathcal{M}, \psi) = (\mathbf{C} \oplus \mathbf{C}, \phi) * (M_2(\mathbf{C}), \tau)$, where τ is the trace-state on the 2×2 matrices and where ϕ is the state on \mathbf{C}^2 assigning to one minimal projection the weight λ , $\frac{1}{2} \leq \lambda < 1$. From [D3] we have that \mathcal{M} is a factor if and only if $\frac{1}{2} \leq \lambda \leq \frac{3}{4}$. However, we have calculated (Examples 3.4i and 3.4iii) that

$$\text{ef}(\mathbf{C}^2, \phi) = \begin{cases} 1 & \text{if } \lambda = \frac{1}{2} \\ 0 & \text{if } \lambda > \frac{1}{2}, \end{cases}$$

and thus only in the case $\lambda = \frac{1}{2}$ does Theorem 2.5 imply that \mathcal{M} is a factor.

- ii) Consider the situation of Example 3.4iv, *i.e.* $A = M_2(\mathbf{C})$ with the state which we will denote ϕ_λ . Figure 1 is a plot of $\text{ef}(A, \phi_\lambda)$ versus λ . We thus find that if $\lambda, \mu \in [\frac{1}{2}, 1)$ and if $(\mathcal{M}, \psi) = (A, \phi_\lambda) * (A, \phi_\mu)$, then Theorem 2.5 implies that \mathcal{M} is a factor for (λ, μ) inside the region bounded by the curve in Figure 2. One might speculate that \mathcal{M} should be factor for all choices of λ and μ , and that Figure 2 illustrates only the limitations of Theorem 2.5.

Examples 4.2. *Type Classification.*

- i) Let (\mathcal{M}, ϕ) and (\mathcal{N}, ψ) be type III von Neumann algebras, each with separable predual. Then their free product von Neumann algebra is a type III factor.
- ii) More generally, if (A, ϕ) and (B, ψ) are any von Neumann algebras having separable preduals and such that ϕ and ψ are normal faithful states, one of which is not a trace, and if A (respectively B) has no minimal projections e such that $\phi(e) > \frac{1}{2}$ (respectively $\psi(e) > \frac{1}{2}$), then the free product von Neumann algebra $A * B$ of $(A * B, \phi * \psi)$ is a type III factor.
- iii) If in the above case we assume that ψ is a trace, then $T(A * B) = I(A, \phi)$.
- iv) If (\mathcal{M}, ϕ) and (\mathcal{N}, ψ) are factors with separable preduals and having types III_λ and III_μ , respectively, with $0 < \lambda, \mu \leq 1$, and if either $\lambda = 1$ or $\mu = 1$, or if $\ln \lambda$ and $\ln \mu$ are not rationally related, then the free product of \mathcal{M} and \mathcal{N} is a factor of type III_1 or type III_0 . If $\ln \lambda$ and $\ln \mu$ are rationally related, then \mathcal{M} is either type III_ν or type III_0 , where $0 < \nu < 1$ is such that $\frac{2\pi}{\ln \nu} \mathbf{Z} = \frac{2\pi}{\ln \lambda} \mathbf{Z} \cap \frac{2\pi}{\ln \mu} \mathbf{Z}$.

Proof. Since, given two von Neumann algebras having separable preduals, their free product von Neumann algebra is faithfully represented on a separable Hilbert space, it too has separable predual. We know from Theorem 2.5 and Lemma 3.2 that the free product von Neumann algebras in the cases given above are factors, so their T -invariants being not \mathbf{R} implies that the factor is type III. To prove iii), note that the modular automorphism group defined with respect to a trace is the identity, so $I(B, \psi) = \mathbf{R}$. In the cases described in part iv, we have clearly that the T -invariant of the free product factor equals $\{0\}$, implying that the factor is type III_1 or III_0 . \square

If \mathcal{M} has semifinite trace Tr and if ϕ is a normal faithful state on \mathcal{M} having density $h \in \mathcal{M}_+$, *i.e.* $\phi(x) = \text{Tr}(hx)$, then ([T], p.98), the modular automorphisms group is given by $\sigma_t^\phi(x) = u(t)xu(-t)$ where $u(t) = h^{it}$. Thus $T(\mathcal{M}) = \mathbf{R}$ and $I(\mathcal{M}, \phi) = \{t \in \mathbf{R} \mid h^{it} \in \mathbf{C}1\}$. For the case of $n \times n$ matrix algebras, let (A, ϕ) be as described in Lemma 3.3, and let Tr be the trace on $M_n(\mathbf{C})$ such that $\text{Tr}(e_{ii}) = 1$. Then the density h is the diagonal matrix with diagonal entries $\lambda_1, \lambda_2, \dots, \lambda_n$. Hence

$$\begin{aligned} I(A, \phi) &= \{t \in \mathbf{R} \mid \left(\frac{\lambda_\iota}{\lambda_j}\right)^{it} = 1 \ \forall \ 1 \leq \iota, j \leq n\} \\ &= \{t \in \mathbf{R} \mid \left(\frac{\lambda_1}{\lambda_\iota}\right)^{it} = 1 \ \forall \ 1 \leq \iota \leq n\}. \end{aligned}$$

So if ϕ is not a trace then

$$I(A, \phi) = \bigcap_{\{2 \leq \iota \leq n \mid \lambda_\iota \neq \lambda_1\}} \left(\frac{2\pi}{\ln \lambda_1 - \ln \lambda_\iota} \right) \mathbf{Z}. \quad (23)$$

To make a connection with F. Rădulescu's work in [R5], for the state ψ_λ on $M_2(\mathbf{C})$ (described in the introduction), we get from (23) that $I(M_2(\mathbf{C}), \psi_\lambda) = \frac{2\pi}{-\ln \lambda} \mathbf{Z}$. Then Theorem 2.5 implies that for $(\mathcal{M}, \phi) = (M_2(\mathbf{C}), \psi_\lambda) * (L(\mathbf{Z}), \tau)$, where τ is a faithful trace on the diffuse Abelian von Neumann algebra $L(\mathbf{Z})$, we have that \mathcal{M} is a type III factor and that $T(\mathcal{M}) = \frac{2\pi}{-\ln \lambda} \mathbf{Z}$, so that \mathcal{M} is either type III_λ or type III_0 . Rădulescu has shown that it is type III_λ and has further described \mathcal{M} in terms of a discrete decomposition for it.

When the conditions of Theorem 2.5 are satisfied for a free product of algebras with states for which we can compute the I -invariants, we can use (23) and (7) to make conclusions about the λ -classification of the free product factor, similarly to what we did in Example 4.2iv. For example, in the notation of the previous paragraph, if $(\mathcal{M}, \phi) = (M_2(\mathbf{C}), \psi_\lambda) * (M_s(\mathbf{C}), \psi_\mu)$, then we can as in Remark 4.1ii decide when the conditions of Theorem 2.5 are satisfied, (note that $\psi_\lambda = \phi_{\lambda'}$, where $\lambda' = \frac{1}{1+\lambda}$). Suppose the conditions of Theorem 2.5 are satisfied. If $\ln \lambda$ and $\ln \mu$ are not rationally related, then \mathcal{M} is either type III_1 or type III_0 . If $\ln \lambda$ and $\ln \mu$ are rationally related, then \mathcal{M} is either type III_ν or type III_0 , where $0 < \nu < 1$ is such that $\frac{2\pi}{\ln \nu} \mathbf{Z} = \frac{2\pi}{\ln \lambda} \mathbf{Z} \cap \frac{2\pi}{\ln \mu} \mathbf{Z}$.

REFERENCES

- [B] Barnett, L., *Free product von Neumann algebras of type III*, preprint, U.C.L.A., 1993.
- [C] Connes, A., *Une classification des facteurs de type III*, Ann. scient. Éc. Norm. Sup. 4^e serie **6** (1973), 133-252.
- [D1] Dykema, K. J., *On certain free product factors via an extended matrix model*, J. Funct. Anal. **112** (1993), 31-60.
- [D2] ———, *Interpolated free group factors*, Pacific J. Math. (to appear).
- [D3] ———, *Free products of hyperfinite von Neumann algebras and free dimension*, Duke Math. J. **69** (1993), 97-119.
- [R1] Rădulescu, F., *The fundamental group of the von Neumann algebra of a free group with infinitely many generators*, J. of the AMS **5** (1992), 517-532.
- [R2] ———, *Stable isomorphism of the weak closure of the group algebras of finite free groups*, I.H.E.S. preprint 1991.
- [R3] ———, *Sous-facteurs de $L(\mathbf{F}_\infty)$ d'indice $4 \cos^2 \pi/n, n \geq 3$* , C. R. Acad. Sci. Paris **t. 315 Série I** (1992), 37-42.
- [R4] ———, *A one parameter group of automorphisms of $L(\mathbf{F}_\infty) \otimes B(\mathfrak{H})$ scaling the trace*, C. R. Acad. Sci. Paris **t. 314 Série I** (1992), 1027-1032.
- [R5] ———, *A type III_λ factor with core isomorphic to the von Neumann algebra of a free group, tensor $B(H)$* , preprint, UC-Berkeley, 1993.
- [S] Strătilă, S., *Modular Theory in Operator Algebras*, Editura Academiei, Bucherest and Abacus Press, England, 1981.
- [SZ] Strătilă, S., Zsido, L., *Lectures on von Neumann Algebras*, Editura Academiei, Bucherest and Abacus Press, England, 1975, 1979.
- [T] Takesaki, M., *Tomita's Theory of Modular Hilbert Algebras and its Applications*, Lecture Notes in Mathematics vol. 128, Springer-Verlag, 1970.
- [V1] Voiculescu, D., *Symmetries of some reduced free product C^* -algebras*, Operator Algebras and Their Connections with Topology and Ergodic Theory, Lecture Notes in Mathematics, Volume 1132, Springer Verlag, 1985, pp. 556-588.
- [V2] ———, *Circular and semicircular systems and free product factors*, Operator Algebras, Unitary Representations, Enveloping Algebras, and Invariant Theory, Progress in Mathematics, Volume 92, Birkhäuser, Boston, 1990.
- [V3] ———, *Limit laws for random matrices and free products*, Inventiones Math. **104** (1991), 201-220.
- [VDN] Voiculescu, D., Dykema, K. J., Nica, A., *Free Random Variables*, CRM Monograph Series vol. 1, American Mathematical Society, 1992.